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AN EQUATION ON OPERATOR ALGEBRAS AND SEMISIMPLE H*−ALGEBRAS

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Abstract. In this paper we prove the following result: Let X be a Banach space over the real or complex field F and let $L(X)$ be the algebra of all bounded linear operators on X. Suppose there exists an additive mapping $T : A(X) \to L(X)$, where $A(X) \subset L(X)$ is a standard operator algebra. Suppose that $T(A^3) = AT(A)A$ holds for all $A \in A(X)$. In this case T is of the form $T(A) = \lambda A$ for any $A \in A(X)$ and some $\lambda \in F$. This result is applied to semisimple H^*−algebras.

This research is related to the work of Molnár [8] and is a continuation of our work [9, 10]. Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, if $nx = 0$, $x \in R$ implies $x = 0$. The commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$, such that $D(x) = [a, x]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [6] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [3]. Cusack [5] generalized Herstein's result to 2−torsion free semiprime rings (see also [2] for an alternative proof).

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An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all $x, y \in R$.

The concept appears naturally in C^*-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a ring module R into itself. For a semiprime ring R all such homomorphisms are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Martindale right ring of quotients Q_r (see Chapter 2 in [2]). In case R has the identity element $T : R \to R$ is a left centralizer iff T is of the form $T(x) = ax$ for all $x \in R$ and some fixed element $a \in R$. An additive mapping $T : R \to R$ is called a left Jordan centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of right centralizer and right Jordan centralizer should be self-explanatory. Following ideas from [4] Zalar [12] has proved that any left (right) Jordan centralizer on a $2-$torsion free semiprime ring is a left (right) centralizer. Molnár [8] has proved that in case we have an additive mapping $T : A \to A$, where A is a semisimple H^*-algebra, satisfying the relation $T(x^3) = T(x)x^2$ $(T(x^3) = x^2T(x))$ for all $x \in A$, then T is a left (right) centralizer. For the definition and for basic facts of H^*-algebras we refer to [1]. Vukman [9] has proved that in case there exists an additive mapping $T : R \longrightarrow R$, where R is a 2-torsion free semiprime ring, satisfying the relation $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$, then$T$is a left and also a right centralizer. Some result concerning centralizers in semiprime rings can be found in [10] and [11]. Let X be a normed space over the real or complex field F, and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subset L(X)$ is said to be standard in case $F(X) \subset A(X)$. Let us point out that any standard algebra is prime, which is a consequence of Hahn-Banach theorem.

We are ready for our first result.

Theorem 1. *Let X be a Banach space over the real or complex field F and let $A(X) \subset L(X)$ be a standard operator algebra. Suppose there exists an additive mapping $T : A(X) \to L(X)$, such that $T(A^3) = AT(A)A$ holds for all $A \in A(X)$. In this case we have $T(A) = \lambda A$ for any $A \in A(X)$ and some $\lambda \in F$.*

Proof. We have the relation

$$T(A^3) = AT(A)A, \text{ for all } A \in A(X). \tag{1}$$

First we will consider the restriction of T on $F(X)$. Let A be from $F(X)$ and let $P \in F(X)$, be a projection such that $AP = PA = A$. From the relation (1) one obtains that $T(P) = PT(P)P$ and $T(P)P = PT(P)$ holds. Putting $A + P$ for A in the relation above and applying the relation (1) we obtain after some calculation

$$\begin{aligned} 3T(A^2) + 3T(A) &= PT(A)A + AT(A)P + AT(P)A \\ &\quad + PT(P)A + AT(P)P + PT(A)P. \end{aligned}$$

Putting $-A$ for A in the above relation and comparing the relation so obtained with the above relation we obtain

$$3T(A^2) = PT(A)A + AT(A)P + AT(P)A, \tag{2}$$

and

$$3T(A) = PT(P)A + AT(P)P + PT(A)P. \tag{3}$$

Multiplying the above relation from both sides by P, we obtain

$$2PT(A)P = PT(P)A + AT(P)P. \tag{4}$$

Combining the relations (3) and (4) we obtain $2T(A) = PT(P)A + AT(P)P$. Now we have $2T(A) = PT(P)A + AT(P)P = (PT(P)P)A + A(PT(P)P) = T(P)A + AT(P)$. Thus we have

$$2T(A) = AB + BA, \tag{5}$$

where B stands for $T(P)$. Now we have $2T(A)P = (AB + BA)P = ABP + BAP = APB + BA = AB + BA = 2T(A)$. We have therefore $T(A)P = T(A)$. Similarly one obtains $PT(A) = T(A)$. Now the relation (2) reduces to

$$3T(A^2) = T(A)A + AT(A) + ABA. \tag{6}$$

Combining (5) and (6) we obtain

$$\begin{aligned} 0 &= 6T(A^2) - 2T(A)A - 2AT(A) - 2ABA \\ &= 3(A^2B + BA^2) - (AB + BA)A - A(AB + BA) - 2ABA \\ &= 2(A^2B + BA^2) - 4ABA. \end{aligned}$$

We have therefore $A^2B + BA^2 = 2ABA$, which can be written according to the relation (5) in the form $T(A^2) = ABA$, which reduces the relation (6) to

$$2T(A^2) = T(A)A + AT(A). \tag{7}$$

The relation (5) makes it possible to concluded that T maps $F(X)$ into itself and that T is linear on $F(X)$. Therefore we have a linear mapping $T : F(X) \to F(X)$ satisfying the relation (7) for all $A \in F(X)$. Since $F(X)$ is prime one can conclude according to Theorem in [9] that T is a left and also a right centralizer. We intend to prove that there exists an operator $C \in L(X)$, such that

$$T(A) = CA, \text{ for all } A \in F(X) \tag{8}$$

For any fixed $x \in X$ and $f \in X^*$ we denote by $x \otimes f$ an operator from $F(X)$ defined by $(x \otimes f)y = f(y)x$, for all $y \in X$. For any $A \in L(X)$ we have $A(x \otimes f) = ((Ax) \otimes f)$. Let us choose f and y such that $f(y) = 1$ and define $Cx = T(x \otimes f)y$. Obviously, C is linear. Using the fact that T is left centralizer on $F(X)$ we obtain

$$\begin{aligned} (CA)x &= C(Ax) = T((Ax) \otimes f)y = T(A(x \otimes f))y \\ &= T(A)(x \otimes f)y = T(A)x, x \in X. \end{aligned}$$

We have therefore $T(A) = CA$ for any $A \in F(X)$. Since T right centralizer on $F(X)$ we obtain $C(AB) = T(AB) = AT(B) = ACB$. We have therefore $[A, C]\, B = 0$ for any $A, B \in F(X)$ whence it follows that $[A, C] = 0$ for any $A \in F(X)$. Using closed graph theorem one can easily prove that C is continuous. Since C commutes with all operators from $F(X)$ one can conclude that $Cx = \lambda x$ holds for any $x \in X$ and some $\lambda \in F$, which gives together with the relation (8) that T is of the form

$$T(A) = \lambda A \tag{9}$$

any $A \in F(X)$ and some $\lambda \in F$. It remains to prove that the above relation holds on $A(X)$ as well. Let us introduce $T_1 : A(X) \to L(X)$ by $T_1(A) = \lambda A$ and consider $T_0 = T - T_1$. The mapping T_0 is, obviously, additive and satisfies the relation (1). Besides, T_0 vanishes on $F(X)$. Let $A \in A(X)$, let P be a one-dimensional projection and $S = A + PAP - (AP + PA)$. Since, obviously, $S - A \in F(X)$, we have $T_0(S) = T_0(A)$. Besides, $SP = PS = 0$. We have therefore the relation

$$T_0(A^3) = AT_0(A)A, \tag{10}$$

for all $A \in A(X)$. Applying the above relation we obtain

$$\begin{aligned} ST_0(S)S &= T_0(S^3) = T_0(S^3 + P) = T_0((S + P)^3) \\ &= (S + P)T_0(S + P)(S + P) = (S + P)T_0(S)(S + P) \\ &= ST_0(S)S + PT_0(S)S + ST_0(S)P + PT_0(S)P. \end{aligned}$$

We have therefore

$$PT_0(A)S + ST_0(A)P + PT_0(A)P = 0. \tag{11}$$

Multiplying the above relation from both sides by P we obtain

$$PT_0(A)P = 0, \tag{12}$$

which reduces the relation (11) to

$$PT_0(A)S + ST_0(A)P = 0. \tag{13}$$

Right multiplication of the above relation by P gives

$$ST_0(A)P = 0. \tag{14}$$

Applying (12) the relation (14) reduces to

$$AT_0(A)P - PAT_0(A)P = 0. \tag{15}$$

Putting in the above relation $A+B$ for A, where A is from $A(X)$ and B from $F(X)$, using the fact that T_0 vanishes on $F(X)$, and applying the relation (15), we obtain

$$0=(A+B)T_0(A)P-P(A+B)T_0(A)P=BT_0(A)P-PBT_0(A)P$$

We have therefore proved that

$$BT_0(A)P-PBT_0(A)P=0$$

holds for any $A\in A(X)$ and all $B\in F(X)$. Putting in the above relation $T_0(A)PB$ for B and applying the relation (12), we obtain

$$(T_0(A)P)B(T_0(A)P)=0,\ \text{for all } B\in F(X),$$

whence it follows $T_0(A)P=0$ by primeness of $F(X)$. Since P is an arbitrary one-dimensional, one can conclude that $T_0(A)=0$, for any $A\in A(X)$, which completes the proof of the theorem. □

In the proof of Theorem 1 we used some ideas from Molnár's paper [8]. Let us point out that in Theorem 1 we obtain as a result the continuity of T under purely algebraic conditions concerning T, which means that Theorem 1 might be of some interest from the automatic continuity point of view.

Theorem 2. *Let A be a semisimple H^*-algebra and let $T:A\rightarrow A$ be such an additive mapping that $T\left(x^3\right)=xT\left(x\right)x$ holds for all $x\in A$. In this case T is a left and a right centralizer.*

Proof. The proof goes through using the same arguments as in the proof of Theorem in [8] with the exception that one has to use Theorem 1 instead of Lemma in [8]. □

Since in the formulation of the theorem above we have used only algebraic concepts, it would be interesting to study the relevant problem in a purely ring theoretical context. Let us point out that Vukman [9] has proved the following result. Let R be a $2-$torsion free semiprime ring and let $T:R\rightarrow R$ be an additive mapping. If $T(xyx)=xT(y)x$ holds for all $x,y\in R$, then T is a left and a right centralizer. In the same paper one can find also a result which states that in case we have a $2-$torsion free semiprime ring with the identity element and an additive mapping $T:R\rightarrow R$ satisfying the relation $T(x^3)=xT(x)x$ for all $x\in R$, then $T(x)=ax$ holds for all $x\in R$ and some $a\in Z(R)$.

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